

In the name of Allah, the Beneficent, the Merciful

AN INVERSION FORMULA FOR MULTIVARIATE POWER SERIES

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Abstract

For formal multivariate power series $\varphi(x)$ an inversion formula of the form

$$\varphi^{-1}(x) = x + \sum_{m=1}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x) \quad \text{is offered}$$

Mathematics Subject Classification: 13F25, 30B10, 16W60.

Key words: multivariate power series, inversion, symmetric product.

1. Introduction

In this paper we are going to show that if $\varphi(x) : F^n \rightarrow F^n$ is a formal power series of the form $\varphi(x) = x +$ "higher order terms" then

$$\varphi^{-1}(x) = x + \sum_{m=1}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x)$$

, where n is any fixed positive integer, F is any field of characteristic zero, $\varphi^{-1}(x)$ stands for formal inversion of φ , \circ stands for composition (superposition) operation, $\varphi^{\circ k}(x)$ stands for k times composition of φ with itself and $\varphi^{\circ 0} = id$ the identity map $id(x) = x$.

Here we are not going to look for the most general case for which this formula is valid and therefore in future one can assume that F is the field of real or complex numbers. To prove this result we need so called "the symmetric product of matrices" which was introduced before, see for example [1], in a little different form.

Here are the needed results, proofs of which (in polynomial case) can be found in [1].

For a positive integer n let I_n stand for all row n -tuples with nonnegative integer entries with the following linear order: $\beta = (\beta_1, \beta_2, \dots, \beta_n) < \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if $|\beta| < |\alpha|$ or $|\beta| = |\alpha|$ and $\beta_1 > \alpha_1$ or $|\beta| = |\alpha|$, $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$, et cetera, where $|\alpha|$ stands for $\alpha_1 + \alpha_2 + \dots + \alpha_n$. We consider in I_n component-wise addition and subtraction (when the result is in I_n), for example, $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$. We write $\beta \ll \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, 2, \dots, n$,

$\binom{\alpha}{\beta}$ stands for $\frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$

For any nonnegative integer numbers p, p' let $M_{n,n}(p', p; F) = M(p', p; F)$ stand for all " $p' \times p$ " size matrices $A = (A_{\alpha}^{\alpha'})_{|\alpha|=p, |\alpha'|=p'}$ (α' presents row, α presents column and $\alpha \in I_n, \alpha' \in I_n$). The ordinary size of a such matrix is $\binom{p' + n - 1}{n - 1} \times \binom{p + n - 1}{n - 1}$. Over such kind matrices in addition to the ordinary sum and product of matrices we consider the following "symmetric product" as well:

Definition 1. If $A \in M(p', p; F)$ and $B \in M(q', q; F)$ then $A \odot B = C \in M(p' + q', p + q; F)$ such that for any $|\alpha| = p + q$, $|\alpha'| = p' + q'$, where $\alpha \in I_n, \alpha' \in I_n$,

$$C_{\alpha}^{\alpha'} = \sum_{\beta, \beta'} \binom{\alpha}{\beta} A_{\beta}^{\beta'} B_{\alpha-\beta}^{\alpha'-\beta'}$$

, where the sum is taken over all $\beta \in I_n, \beta' \in I_n$, for which $|\beta| = p, |\beta'| = p', \beta \ll \alpha$ and $\beta' \ll \alpha'$.

Proposition 1. For the above defined product the following are true.

1. $A \odot B = B \odot A$.
2. $(A + B) \odot C = A \odot C + B \odot C$.
3. $(A \odot B) \odot C = A \odot (B \odot C)$
4. $(\lambda A) \odot B = \lambda(A \odot B)$ for any $\lambda \in F$
5. $A \odot B = 0$ if and only if $A = 0$ or $B = 0$.

In future $A^{\odot m}$ means the m -th power of matrix A with respect to the new product.

Proposition 2. If $h = (h_1, h_2, \dots, h_n) \in M(0, 1; R)$, then for any $|\alpha| = m$

$$(h^{\odot m})_{\alpha}^0 = m!h^{\alpha}$$

, where h^{α} stands for $h_1^{\alpha_1}h_2^{\alpha_2}\dots h_n^{\alpha_n}$

Proposition 3. For any nonnegative integers p, q, p', q' and matrices $A \in M_{n,n}(p', p; F)$, $B \in M_{n,n}(q', q; F)$, $h = (h_1, h_2, \dots, h_n) \in M_{n,n}(0, 1; F)$, the following equality

$$\left(\frac{h^{\odot p}}{p!}A\right) \odot \left(\frac{h^{\odot q}}{q!}B\right) = \frac{h^{\odot(p+q)}}{(p+q)!}(A \odot B)$$

is true.

In future $Mat_{n,n}(F) = Mat(F)$ stands for the set of all block matrices $A = (A(p', p))_{p', p}$ with blocks $A(p', p) \in M_{n,n}(p', p; F)$ for all nonnegative integers p, p' . In future it is assumed that $M(p', p; F)$ is a subset of $Mat(F)$ by identifying each $A(p', p) \in M(p', p; F)$ as the element of $Mat(F)$ which's all blocks are zero, may be, except for (p', p) block which is $A(p', p)$.

For any $A, B \in Mat(F)$ we define $A \odot B = C \in Mat(F)$, where for all nonnegative integers p, p'

$$C(p', p) = \sum_{q', q} A(q', q) \odot B(p' - q', p - q)$$

The above Propositions show that $(Mat(F); +, \odot)$ is an integral domain. Its identity element will be $1 \in Mat(F)$ whose all blocks are zero except for $(0, 0)$ block which is 1 - the identity element of F , F is a subring of $Mat(F)$.

In future the expression $e^{\odot A}$, whenever it has meaning, stands for

$$1 + \frac{1}{1!}A + \frac{1}{2!}A^{\odot 2} + \frac{1}{3!}A^{\odot 3} + \dots = \sum_{i=0}^{\infty} \frac{1}{i!}A^{\odot i}$$

, $F[[x]]$ is the ring of formal power series in variables x_1, x_2, \dots, x_n over F , $x = (x_1, x_2, \dots, x_n) \in M_{n,n}(0, 1; F[[x]])$.

If $\varphi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)) \in F[[x]]$ one can screen it in the form

$$\varphi(x) = x^{\odot 0}M_1^0 + \frac{x^{\odot 1}}{1!}M_1^1 + \frac{x^{\odot 2}}{2!}M_1^2 + \dots = \sum_{i=0}^{\infty} \frac{x^{\odot i}}{i!}M_1^i = e^{\odot x}M_{\varphi}$$

, where $x^{\odot 0} = 1$, $M_1^{p'} \in Mat(p', 1; F)$, $M_{\varphi} \in Mat(F)$ with blocks $M_{\varphi}(p, p')$ such that $M_{\varphi}(p', p) = 0$ whenever $p \neq 1$ and $M_{\varphi}(p', 1) = M_1^{p'}$ for all nonnegative integers. We call M_{φ} the matrix of $\varphi(x)$.

Let us consider only power series with zero constant terms.

The following theorem deals with the matrix of composition of power series.

Theorem 1. If $\psi(x) = (\psi_1(x), \psi_2(x), \dots, \psi_n(x)) = e^{\odot x}M_{\psi} \in F[[x]]$ then

$$M_{\psi \circ \varphi} = e^{\odot M_{\varphi}}M_{\psi}$$

The associative property of composition yields in the following result.

Theorem 2. If $\xi(x) = (\xi_1(x), \xi_2(x), \dots, \xi_n(x)) = e^{\odot x} M_\xi \in F[[x]]$ then

$$e^{\odot M_\varphi} (e^{\odot M_\psi} M_\xi) = e^{\odot (e^{\odot M_\varphi} M_\psi)} M_\xi$$

Corollary 1. For any natural m one has

$$M_{\varphi^{\circ m}} = (e^{\odot M_\varphi})^{m-1} M_\varphi = (e^{\odot M_\varphi})^m E_1$$

, where E_1 is " 1×1 " size identity matrix.

2. The main result

In future we consider any fixed $\varphi(x) = e^{\odot x} M_\varphi$ for which $M_\varphi(0, 1) = 0$, $M_\varphi(1, 1) = E_1$, where $M_\varphi(p', 1) = M_1^{p'} \in \text{Mat}(p', 1; F)$ are arbitrary for $p' \geq 2$. Let $\varphi^{-1}(x) = e^{\odot x} M_{\varphi^{-1}}$ stand for inverse formal power series to φ . The block components of $M_{\varphi^{-1}}$ we denote by $N_1^{p'}$

Theorem 3. For the above mentioned power series $\varphi(x)$ the following equality

$$\varphi^{-1}(x) = \sum_{m=0}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x) = x + \sum_{m=1}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x)$$

is true.

Proof. Due to Theorem 1 equality $\varphi^{-1}(\varphi(x)) = x$ is nothing than $e^{\odot M_\varphi} M_{\varphi^{-1}} = E_1$ and therefore $M_{\varphi^{-1}} = (e^{\odot M_\varphi})^{-1} E_1$ provided that $e^{\odot M_\varphi}$ is invertible.

But in our case

$$(e^{\odot M_\varphi})^{-1} = (E_\infty - (E_\infty - e^{\odot M_\varphi}))^{-1} = E_\infty + \sum_{m=1}^{\infty} (E_\infty - e^{\odot M_\varphi})^m$$

is well defined, where E_∞ stands for infinite size identity matrix.

Due to Corollary 1 one can see that

$$(E_\infty - e^{\odot M_\varphi})^m E_1 = \sum_{k=0}^m (-1)^k \binom{m}{k} M_{\varphi^{\circ k}}$$

, where $\varphi^{\circ 0} = id$ -the identity map $id(x) = x$.

It implies that

$$M_{\varphi^{-1}} = E_1 + \sum_{m=1}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} M_{\varphi^{\circ k}}$$

Now one can write this result in terms of φ

$$\varphi^{-1}(x) = x + \sum_{m=1}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x)$$

or in a symbolic form

$$\varphi^{-1}(x) = x + \sum_{m=1}^{\infty} (id - \varphi)^{[\odot]m}(x)$$

, where $(id - \varphi)^{[\odot]m}(x)$ stands for $\sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x)$, that is one can remove parentheses in $(id - \varphi)^{[\odot]m}(x)$ as if φ were a linear operator. This is the proof of Theorem 3.

One can ask the following question.

Question. If both of $\varphi(x)$, $\varphi^{-1}(x)$ are polynomial maps does it imply that for some $m_0 > 0$

$$\sum_{m=m_0}^{\infty} \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x) = 0 ?$$

In common case I am not sure that the answer to this question is positive.

Due to

$$\begin{aligned} & \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} (e^{\odot M_\varphi})^k E_1 = (E_\infty - e^{\odot M_\varphi})^{m+1} E_1 = (E_\infty - e^{\odot M_\varphi})^m M_{id-\varphi} = \\ & \sum_{k=0}^m (-1)^k \binom{m}{k} (e^{\odot M_\varphi})^k M_{id-\varphi} = \sum_{k=0}^m (-1)^k \binom{m}{k} M_{(id-\varphi) \circ \varphi^{\circ k}} = \sum_{k=0}^m (-1)^k \binom{m}{k} (M_{\varphi^{\circ k}} - M_{\varphi^{\circ(k+1)}}) \end{aligned}$$

for $\Phi_m(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x)$ one has $\Phi_{m+1}(x) = \Phi_m(x) - \Phi_m(\varphi(x))$, where $\Phi_0 = id$.

Proposition 4. For any $m_0 \geq 0$ the following equality

$$\Phi_{m_0}(x) = \sum_{m=m_0}^{\infty} (\Phi_m \circ \varphi)(x) = \sum_{m=m_0}^{\infty} \Phi_m(\varphi(x))$$

is true.

Proof. At $m_0 > 0$ due to the equality $\Phi_{m+1}(x) = \Phi_m(x) - \Phi_m(\varphi(x))$ one has

$$\sum_{m=m_0}^{\infty} \Phi_m(x) = \sum_{m=m_0}^{\infty} (\Phi_{m-1}(x) - \Phi_{m-1}(\varphi(x))) =$$

$$\Phi_{m_0-1}(x) - \Phi_{m_0-1}(\varphi(x)) + \sum_{m=m_0+1}^{\infty} (\Phi_{m-1}(x) - \Phi_{m-1}(\varphi(x))) = \Phi_{m_0}(x) + \sum_{m=m_0}^{\infty} \Phi_m(x) - \sum_{m=m_0}^{\infty} \Phi_m(\varphi(x))$$

, which implies that $\Phi_{m_0}(x) = \sum_{m=m_0}^{\infty} \Phi_m(\varphi(x))$. The case $m_0 = 0$ is a consequence of the case $m_0 = 1$.

Corollary 2. If $\sum_{m=m_0}^{\infty} \Phi_m(x) = 0$ for some $m_0 \geq 1$ then $\Phi_m(x) = 0$ for any $m \geq m_0$.

Corollary 3. If $\varphi(x)$ is a polynomial map and for some $m \geq 1$ the equality $\sum_{k=0}^m (-1)^k \binom{m}{k} \varphi^{\circ k}(x) = 0$ is true

then $\varphi^{-1}(x)$ is also a polynomial map.

In an equivalent form the condition of Corollary 3 can be given in the form: For some $m \geq 1$ the equality

$$(\partial\varphi)(x)(mE_1 + \sum_{k=2}^m (-1)^{k-1} \binom{m}{k} (\partial\varphi)|_{\varphi(x)} (\partial\varphi)|_{\varphi^{\circ 2}(x)} \dots (\partial\varphi)|_{\varphi^{\circ(k-1)}(x)}) = E_1$$

is true, where $\partial_k = \frac{\partial}{\partial x_k}$, $(\partial\varphi(x))_j^i = \partial_i \varphi_j(x)$, $(\partial\varphi)|_{\varphi^{\circ(k-1)}(x)} = (\partial\varphi)(\varphi^{\circ(k-1)}(x))$.

For block components of $M_{\varphi^{-1}}$ the following recurrent formula is true.

Proposition 5. For any $m > 1$ one has

$$N_1^m = - \sum_{k=1}^{m-1} \left(\sum_{|\alpha|=k, \|\alpha\|=m} \frac{(M_1^1)^{\odot \alpha_1} \odot (M_1^2)^{\odot \alpha_2} \odot (M_1^3)^{\odot \alpha_3} \odot \dots}{\alpha!} \right) N_1^k$$

and $N_1^1 = M_1^1$, where $\|\alpha\| = 1\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots$

Proof. The above equalities are nothing than the equality $e^{\odot M_\varphi} M_{\varphi^{-1}} = E_1$ in $(m, 1)$ blocks for $m > 1$.

Remark. The set of polynomial maps $\varphi(x)$ for which $\varphi^{-1}(x)$ is also polynomial map is a group with respect to the composition operation. Finding any system of generators of it may be useful.

References

[1] Ural Bekbaev. *A matrix representation of composition of polynomial maps.*

arXiv0901.3179v3 [math. AC] 22 Sep.2009